11.7.2Asset Driven by a Brownian Motion and a Compound Poisson Process

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Theorem 11.7.5. *For $0 \le t < T$, the risk-neutral price of a call,*

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ \middle| \mathcal{F}(t)\right],$$

is given by $V(t) = c(t, S(t))$, where

$$c(t,x) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)} \frac{\tilde{\lambda}^j (T-t)^j}{j!} \widetilde{\mathbb{E}}\,\kappa\left(T-t, xe^{-\tilde{\beta}\tilde{\lambda}(T-t)} \prod_{i=1}^{j}(Y_i+1)\right). \quad (11.7.30)$$

$$S(t) = S(0)\exp\left\{\sigma\widetilde{W}(t) + \left(r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2\right)t\right\} \prod_{i=1}^{N(t)}(Y_i+1). \qquad (11.7.28)$$

PROOF: Let $t \in [0,T)$ be given and define $\tau = T - t$. From (11.7.28), we see that

$$S(T) = S(t)\exp\left\{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + \left(r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2\right)\tau\right\} \prod_{i=N(t)+1}^{N(T)}(Y_i+1). \quad (11.7.31)$$

The term $S(t)$ is $\mathcal{F}(t)$-measurable, and the other term appearing on the right-hand side of (11.7.31) is independent of $\mathcal{F}(t)$. Therefore, the Independence Lemma, Lemma 2.3.4, implies that

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r\tau}(S(T)-K)^+ \middle| \mathcal{F}(t)\right] = c(t, S(t)),$$

Lemma 2.3.4 (Independence). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose the random variables $X_1, \ldots, X_K$ are $\mathcal{G}$-measurable and the random variables $Y_1, \ldots, Y_L$ are independent of $\mathcal{G}$. Let $f(x_1, \ldots, x_K, y_1, \ldots, y_L)$ be a function of the dummy variables $x_1, \ldots, x_K$ and $y_1, \ldots, y_L$, and define*

$$g(x_1, \ldots, x_K) = \mathbb{E} f(x_1, \ldots, x_K, Y_1, \ldots, Y_L). \qquad (2.3.27)$$

Then

$$\mathbb{E}[f(X_1, \ldots, X_K, Y_1, \ldots, Y_L) | \mathcal{G}] = g(X_1, \ldots, X_K). \qquad (2.3.28)$$

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r\tau}(S(T) - K)^+ \middle| \mathcal{F}(t)\right] = c(t, S(t)),$$

where

$$\begin{aligned} c(t,x) &= \widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x \exp\left\{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + \left(r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2\right)\tau\right\} \right.\right. \\ &\qquad \left.\left. \times \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K\right)^+\right] \end{aligned}$$

$$S(T) = S(t)\exp\left\{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + \left(r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2\right)\tau\right\} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1). \quad (11.7.31)$$

Law of total expectation E(X)=E(E(X|Y))

$$\begin{aligned} &= \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x \exp\left\{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + \left(r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2\right)\tau\right\} \right.\right.\right. \\ &\qquad \left.\left.\left. \times \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K\right)^+ \middle| \sigma\left(\prod_{i=N(t)+1}^{N(T)} (Y_i + 1)\right)\right]\right] \end{aligned}$$

where the conditioning σ-algebra $\sigma\left(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\right)$ is the one generated by the random variable $\prod_{i=N(t)+1}^{N(T)}(Y_i+1)$.

$$V(t)=\widetilde{\mathbb{E}}\left[e^{-r\tau}\left(S(T)-K\right)^{+}\middle|\mathcal{F}(t)\right]=c(t,S(t)),$$

where

$$\begin{aligned}
c(t,x) & \\
&=\widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x\exp\left\{\sigma(\widetilde{W}(T)-\widetilde{W}(t))+\left(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2\right)\tau\right\}\right.\right.\\
&\qquad\left.\left.\times\prod_{i=N(t)+1}^{N(T)}(Y_i+1)-K\right)^{+}\right]\\
&=\widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x\exp\left\{\sigma(\widetilde{W}(T)-\widetilde{W}(t))+\left(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2\right)\tau\right\}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\prod_{i=N(t)+1}^{N(T)}(Y_i+1)-K\right)^{+}\middle|\sigma\left(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\right)\right]\right]\\
&=\widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[e^{-r\tau}\left(xe^{-\tilde{\beta}\tilde{\lambda}\tau}\exp\left\{-\sigma\sqrt{\tau}Y+\left(r-\frac{1}{2}\sigma^2\right)\tau\right\}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\prod_{i=N(t)+1}^{N(T)}(Y_i+1)-K\right)^{+}\middle|\sigma\left(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\right)\right]\right],
\end{aligned}$$

$$S(T)=S(t)\exp\left\{\sigma(\widetilde{W}(T)-\widetilde{W}(t))+\left(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2\right)\tau\right\}\prod_{i=N(t)+1}^{N(T)}(Y_i+1). \quad (11.7.31)$$

Law of total expectation E(X)=E(E(X|Y))

where $Y=-\dfrac{\widetilde{W}(T)-\widetilde{W}(t)}{\sqrt{\tau}}$

is a standard normal random variable under $\widetilde{\mathbb{P}}$

$W(T)-W(t)\sim N(0,T-t)$

$$V(t) = \widetilde{\mathbb{E}}[e^{-r\tau}(S(T)-K)^+ | \mathcal{F}(t)] = c(t, S(t)),$$

where

$$\begin{aligned} c(t,x) &= \widetilde{\mathbb{E}}\Bigg[\widetilde{\mathbb{E}}\Bigg[e^{-r\tau}\Bigg(xe^{-\tilde{\beta}\tilde{\lambda}\tau}\exp\Big\{-\sigma\sqrt{\tau}Y + \Big(r-\frac{1}{2}\sigma^2\Big)\tau\Big\} \\ &\qquad \times \prod_{i=N(t)+1}^{N(T)}(Y_i+1) - K\Bigg)^+ \Bigg| \sigma\Bigg(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\Bigg)\Bigg]\Bigg], \end{aligned}$$

Because $\prod_{i=N(t)+1}^{N(T)}(Y_i+1)$ is $\sigma\big(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\big)$-measurable and Y is independent of $\sigma\big(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\big)$, we may use the Independence Lemma, Lemma 2.3.4, again to obtain

$$\begin{aligned} &\widetilde{\mathbb{E}}\Bigg[e^{-r\tau}\Bigg(xe^{-\tilde{\beta}\tilde{\lambda}}\exp\Big\{-\sigma\sqrt{\tau}Y + \Big(r-\frac{1}{2}\sigma^2\Big)\tau\Big\} \\ &\qquad \times \prod_{i=N(t)+1}^{N(T)}(Y_i+1) - K\Bigg)^+ \Bigg| \sigma\Bigg(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\Bigg)\Bigg] \\ &\qquad = \kappa\Bigg(\tau, xe^{-\tilde{\beta}\tilde{\lambda}\tau}\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\Bigg). \end{aligned}$$

Lemma 2.3.4 (Independence). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose the random variables $X_1, \dots, X_K$ are $\mathcal{G}$-measurable and the random variables $Y_1, \dots, Y_L$ are independent of $\mathcal{G}$. Let $f(x_1, \dots, x_K, y_1, \dots, y_L)$ be a function of the dummy variables $x_1, \dots, x_K$ and $y_1, \dots, y_L$, and define*

$$g(x_1, \dots, x_K) = \mathbb{E}f(x_1, \dots, x_K, Y_1, \dots, Y_L). \quad (2.3.27)$$

Then

$$\mathbb{E}[f(X_1, \dots, X_K, Y_1, \dots, Y_L)|\mathcal{G}] = g(X_1, \dots, X_K). \quad (2.3.28)$$

$$c(t,x) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)} \frac{\tilde{\lambda}^j (T-t)^j}{j!} \widetilde{\mathbb{E}} \kappa\Big(T-t, xe^{-\tilde{\beta}\tilde{\lambda}(T-t)} \prod_{i=1}^{j} (Y_i+1)\Big). \quad (11.7.30)$$

It follows that

$$c(t,x) = \mathbb{E}\,\kappa\left(\tau, xe^{-\tilde{\beta}\tilde{\lambda}\tau} \prod_{i=N(t)+1}^{N(T)} (Y_i+1)\right). \qquad (11.7.32)$$

To see that (11.7.32) agrees with (11.7.30), we note that conditioned on $N(T) - N(t) = j$, the random variable $\prod_{i=N(t)+1}^{N(T)}(Y_i+1)$ has the same distribution as $\prod_{i=1}^{j}(Y_i+1)$. Furthermore,

$$\mathbb{P}\{N(T) - N(t) = j\} = e^{-\tilde{\lambda}\tau} \frac{\tilde{\lambda}^j \tau^j}{j!}.$$

□

Let $0 \le s < t$ be given. According to Theorem 11.2.3, the Poisson increment $N(t) - N(s)$ has distribution

$$\mathbb{P}\{N(t) - N(s) = k\} = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}, \quad k = 0, 1, \ldots. \qquad (11.2.8)$$

Remark 11.7.6 (Continuous jump distribution). Suppose the jump sizes Y_i have a density $f(y)$ rather than a probability mass function $p(y_1), \ldots, p(y_m)$, and this density is strictly positive on a set $B \subset (-1, \infty)$ and zero elsewhere. In this case, we replace (11.7.17) by the formula

$$\beta = \mathbb{E}Y_i = \int_{-1}^{\infty} y f(y)\, dy.$$

For the risk-neutral measure, we can choose θ, $\tilde{\lambda} > 0$ and any density $\widetilde{f}(y)$ that is strictly positive on B and zero elsewhere so that the market price of risk equation (see (11.7.26))

$$\alpha - r = \sigma\theta + \beta\lambda - \tilde{\beta}\tilde{\lambda}$$

is satisfied, where now

$$\tilde{\beta} = \widetilde{\mathbb{E}}Y_i = \int_{-1}^{\infty} y \widetilde{f}(y)\, dy.$$

Define

$$p(y_m) = \frac{\lambda_m}{\lambda}.$$

The random variables $Y_1, Y_2, \ldots$ are independent and identically distributed, with $\mathbb{P}\{Y_i = y_m\} = p(y_m)$. These assertions all follow from Theorem 11.3.3.

Set

$$\beta = \mathbb{E}Y_i = \sum_{m=1}^{M} y_m p(y_m) = \frac{1}{\lambda} \sum_{m=1}^{M} \lambda_m y_m. \qquad (11.7.17)$$

We return to the model with discrete jump sizes. The following theorem provides the differential-difference equation satisfied by the call price.

Theorem 11.7.7. *The call price $c(t,x)$ of (11.7.30) satisfies the equation*

$$-rc(t,x) + c_t(t,x) + (r - \tilde{\beta}\tilde{\lambda})xc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x)$$
$$+\tilde{\lambda}\Big[\sum_{m=1}^{M} \tilde{p}(y_m)c(t,(y_m+1)x) - c(t,x)\Big] = 0,\ \ 0 \le t < T,\ x \ge 0,\ (11.7.33)$$

and the terminal condition

$$c(T,x) = (x-K)^+,\ \ x \ge 0.$$

$$c(t,x) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)} \frac{\tilde{\lambda}^j (T-t)^j}{j!} \widetilde{\mathbb{E}}\,\kappa\Big(T-t, xe^{-\tilde{\beta}\tilde{\lambda}(T-t)} \prod_{i=1}^{j}(Y_i+1)\Big).\ \ (11.7.30)$$

PROOF: From (11.7.27), we see that the continuous part of the stock price satisfies $dS^c(t) = (r - \tilde{\beta}\tilde{\lambda})S(t)\,dt + \sigma S(t)\,d\widetilde{W}(t)$. Therefore, the Itô-Doeblin formula implies

$$
\begin{aligned}
&e^{-rt}c(t, S(t)) - c(0, S(0)) \\
&= \int_0^t e^{-ru}\Big[-rc(u, S(u)) + c_t(u, S(u)) + (r - \tilde{\beta}\tilde{\lambda})S(u)c_x(u, S(u)) \\
&\qquad + \frac{1}{2}\sigma^2 S^2(u)c_{xx}(u, S(u))\Big]\,du + \int_0^t e^{-ru}\sigma S(u)c_x(u, S(u))\,d\widetilde{W}(u) \\
&\quad + \sum_{0<u\leq t} e^{-ru}\big[c(u, S(u)) - c(u, S(u-))\big]. \qquad (11.7.34)
\end{aligned}
$$

$$
\begin{aligned}
dS(t) &= rS(t)dt + \sigma S(t)d\widetilde{W}(t) + S(t-)d\big(Q(t) - \tilde{\beta}\tilde{\lambda}t\big) \\
&= \big(r - \tilde{\beta}\tilde{\lambda}\big)S(t)dt + \sigma S(t)d\widetilde{W}(t) + S(t-)dQ(t)
\end{aligned}
$$

$$
\begin{aligned}
&e^{-rt}c(t,S(t)) - c(0,S(0)) \\
&= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde{\beta}\tilde{\lambda})S(u)c_x(u,S(u)) \\
&\qquad + \frac{1}{2}\sigma^2 S^2(u) c_{xx}(u,S(u))\Big]\,du + \int_0^t e^{-ru}\sigma S(u)c_x(u,S(u))\,d\widetilde{W}(u) \\
&\qquad + \sum_{0<u\le t} e^{-ru}\big[c(u,S(u)) - c(u,S(u-))\big]. \qquad (11.7.34)
\end{aligned}
$$

Theorem 11.5.4 (Two-dimensional Itô-Doeblin formula for processes with jumps). *Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t,x_1,x_2)$ be a function whose first and second partial derivatives appearing in the following formula are defined and are continuous. Then*

$$
\begin{aligned}
&f(t,X_1(t),X_2(t)) \\
&= f(0,X_1(0),X_2(0)) + \int_0^t f_t(s,X_1(s),X_2(s))\,ds \\
&\quad + \int_0^t f_{x_1}(s,X_1(s),X_2(s))\,dX_1^c(s) + \int_0^t f_{x_2}(s,X_1(s),X_2(s))\,dX_2^c(s) \\
&\quad + \frac{1}{2}\int_0^t f_{x_1,x_1}(s,X_1(s),X_2(s))\,dX_1^c(s)\,dX_1^c(s) \\
&\quad + \int_0^t f_{x_1,x_2}(s,X_1(s),X_2(s))\,dX_1^c(s)\,dX_2^c(s) \\
&\quad + \frac{1}{2}\int_0^t f_{x_2,x_2}(s,X_1(s),X_2(s))\,dX_2^c(s)\,dX_2^c(s) \\
&\quad + \sum_{0<s\le t}\big[f(s,X_1(s),X_2(s)) - f(s,X_1(s-),X_2(s-))\big].
\end{aligned}
$$

$dS^c(t) = (r-\tilde{\beta}\tilde{\lambda})S(t)dt + \sigma S(t)d\tilde{W}(t)$

$dS^c(t)dS^c(t) = \sigma^2 S^2(t)dt$

$f(t,X_1(t)) = e^{-rt}c(t,S(t))$

$f(0,X_1(0)) = c(0,S(0))$

$f_t(s,X_1(s))ds = [-re^{-ru}c(u,S(u)) + e^{-ru}c_t(u,S(u))]du$

$f_{x_1}(s,X_1(s))dX_1^c(s) = e^{-ru}c_{x_1}(u,S(u))dS^c(u)$

$f_{x_1x_1}(s,X_1(s))dX_1^c(s)dX_1^c(s) = e^{-ru}c_{x_1x_1}(u,S(u))dS^c(u)dS^c(u)$

$\Sigma[f(s,X_1(s)) - f(s,X_1(s-))] = \Sigma[e^{-ru}c(u,S(u)) - e^{-ru}c(u,S(u-))]$

We examine the last term in (11.7.34). If u is a jump time of the mth Poisson process N_m, the stock price satisfies $S(u) = (y_m+1)S(u-)$. Therefore,

$$\sum_{0<u\le t} e^{-ru}\big[c(u,S(u)) - c(u,S(u-))\big]$$

$$= \sum_{m=1}^{M}\sum_{0<u\le t} e^{-ru}\big[c(u,(y_m+1)S(u-)) - c(u,S(u-))\big]\,\Delta N_m(u)$$

$$= \sum_{m=1}^{M}\int_0^t e^{-ru}\big[c(u,(y_m+1)S(u-)) - c(u,S(u-))\big]\,d(N_m(u)-\tilde\lambda_m u)$$

$$+\int_0^t e^{-ru}\Big[\sum_{m=1}^{M}\frac{\tilde\lambda_m}{\tilde\lambda}c(u,(y_m+1)S(u)) - c(u,S(u))\Big]\tilde\lambda\,du$$

$$= \sum_{m=1}^{M}\int_0^t e^{-ru}\big[c(u,(y_m+1)S(u-)) - c(u,S(u-))\big]\,d(N_m(u)-\tilde\lambda_m u)$$

$$+\int_0^t e^{-ru}\tilde\lambda\sum_{m=1}^{M}\big[\tilde p(y_m)c(u,(y_m+1)S(u)) - c(u,S(u))\big]\Big\}\,du.$$

$$\tilde p(y_m) = \frac{\tilde\lambda_m}{\tilde\lambda}$$

Substituting this into (11.7.34) and taking differentials, we obtain

$$
\begin{aligned}
d\left(e^{-rt}c(t,S(t))\right)
&= e^{-rt}\Big\{-rc(t,S(t)) + c_t(t,S(t)) + (r-\tilde\beta\tilde\lambda)S(t)c_x(t,S(t)) \\
&\qquad + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)) \\
&\qquad + \tilde\lambda \sum_{m=1}^{M}\left[\tilde p(y_m)c(t,(y_m+1)S(t)) - c(t,S(t))\right]\Big\}\,dt \\
&\quad + e^{-rt}\sigma S(t)c_x(t,S(t))\,d\widetilde W(t) \\
&\quad + \sum_{m=1}^{M} e^{-rt}\left[c(t,(y_m+1)S(t-)) - c(t,S(t-))\right]d(N_m(t)-\tilde\lambda_m t). \quad (11.7.35)
\end{aligned}
$$

martingale

$$
\begin{aligned}
&e^{-rt}c(t,S(t)) - c(0,S(0)) \\
&= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde\beta\tilde\lambda)S(u)c_x(u,S(u)) \\
&\qquad + \frac{1}{2}\sigma^2 S^2(u)c_{xx}(u,S(u))\Big]du + \int_0^t e^{-ru}\sigma S(u)c_x(u,S(u))\,d\widetilde W(u) \\
&\quad + \sum_{0<u\le t} e^{-ru}\left[c(u,S(u)) - c(u,S(u-))\right]. \qquad (11.7.34)
\end{aligned}
$$

last term in (11.7.34)

$$
\begin{aligned}
&= \sum_{m=1}^{M}\int_0^t e^{-ru}\left[c(u,(y_m+1)S(u-)) - c(u,S(u-))\right]d(N_m(u)-\tilde\lambda_m u) \\
&\quad + \int_0^t e^{-ru}\tilde\lambda\sum_{m=1}^{M}\left[\tilde p(y_m)c(u,(y_m+1)S(u)) - c(u,S(u))\right]\Big\}\,du.
\end{aligned}
$$

The integrators $N_m(t)-\lambda_m t$ in the last term are martingales under $\mathbb{P}$, and the integrands $e^{-rt}[c(t,(y_m+1)S(t-)) - c(t,S(t-))]$ are left-continuous. Therefore, the integral of this term is a martingale. Likewise, the integral of the next-to-last term $e^{-rt}c_x(t,S(t))\,d\widetilde W(t)$ is a martingale. Since the discounted option price appearing on the left-hand side of (11.7.35) is also a martingale, the remaining term in (11.7.35) is a martingale as well. Because the remaining term is a dt term, it must be zero. Replacing the price process $S(t)$ by the dummy variable x in the integrand of this term, we obtain (11.7.33). □

martingale没有drift term

Theorem 11.4.5. *Assume that the jump process $X(s)$ of (11.4.1)–(11.4.3) is a martingale, the integrand $\Phi(s)$ is* left-continuous *and adapted, and*

$$\mathbb{E}\int_0^t \Gamma^2(s)\Phi^2(s)\,ds < \infty \text{ for all } t \ge 0.$$

Then the stochastic integral $\int_0^t \Phi(s)\,dX(s)$ is also a martingale.

$$
\begin{aligned}
&-rc(t,x) + c_t(t,x) + (r-\tilde\beta\tilde\lambda)xc_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x) \\
&\quad + \tilde\lambda\left[\sum_{m=1}^{M}\tilde p(y_m)c(t,(y_m+1)x) - c(t,x)\right] = 0,\ \ 0\le t<T,\ x\ge 0, \quad (11.7.33)
\end{aligned}
$$